A NOTE ON THE MOMENT MAP ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT. We consider compact Kähler manifolds acted on by a connected compact Lie group K of isometries in a Hamiltonian fashion. We prove that the squared moment map $||\mu||^2$ is constant if and only if the manifold is biholomorphically and K-equivariantly isometric to a product of a flag manifold and a compact Kähler manifold which is acted on trivially by K.

The authors do not know whether the compactness of M is essential in the main theorem; more generally it would be interesting to have a similar result for (compact) symplectic manifolds.

1. INTRODUCTION

We shall consider compact Kähler manifolds M acted on by a compact connected Lie group K of isometries; such isometries are automatically holomorphic transformations of M. We shall also suppose that the K-action on M is Hamiltonian, i.e. there exists a moment map $\mu : M \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K; such a moment map exists if and only if the Lie group K acts trivially on the Albanese torus $Alb(M)$ (see [HW],[GS]). Throughout the following we will denote by g, J and ω the Riemannian metric, the complex structure and the corresponding Kähler form on M respectively; moreover Lie groups and their Lie algebras will be indicated with capital and gothic letters respectively.

If we fix an $Ad(K)$-invariant scalar product $q := \langle , \rangle$ on $\mathfrak{k}$ and we identify $\mathfrak{k}^*$ with $\mathfrak{k}$ by means of q, we can think of μ as a $\mathfrak{k}$-valued map ;the function $f \in C^\infty(M)$ defined as $f := ||\mu||^2$ has been extensively used in [K1] to obtain strong information on the topology of the manifold.

Our main result is the following

Theorem 1. *Suppose M is a compact Kähler K-Hamiltonian manifold, where K is a compact connected Lie group of isometries. If μ denotes the moment map, then the squared moment map $f = ||\mu||^2$ is constant if and only if the manifold M is biholomorphically and K-equivariantly isometric to the product of a flag manifold and a compact Kähler manifold which is acted on trivially by K.*

In order to prove the above theorem, we need the following result, which has been proved in [GP].

Proposition 2. *Let M be a compact Kähler manifold which is acted on by a compact connected Lie group K of isometries in a Hamiltonian fashion with moment map μ. If a point $x \in M$ realizes the maximum of $||\mu||^2$, then the orbit $K \cdot x$ is complex.*

The authors do not know whether the compactness of M is essential in the main theorem; more generally it would be interesting to have a similar result for (compact) symplectic manifolds.

2. Proof of the main result

For later use, we reproduce here the proof of Proposition 2.

Proof of Proposition 2. We will follow the notations as in [K1]. Let $\beta = \mu(x)$, which we can suppose to lie in the closure of a Weyl chamber $\mathfrak{t}_+$, where $\mathfrak{t}$ denotes the Lie algebra of a fixed maximal torus in K. If $\mu_\beta := \langle \mu, \beta \rangle$ is the height function relative to β and if Z_β denotes the union of the connected components of the critical point set of μ_β on which μ_β takes the value $||\beta||^2$, then x belongs to the critical set $C_\beta = K \cdot (Z_\beta \cap \mu^{-1}(\beta))$. We now claim that $Z_\beta = \mu^{-1}(\beta)$. Indeed, if $p \in Z_\beta$, then $\mu_\beta(p) = ||\beta||^2$ and

$$||\beta||^2 \le \langle \mu(p), \beta \rangle \le ||\mu(p)|| \cdot ||\beta|| \le ||\beta||^2,$$

and therefore $\mu(p) = \beta$, i.e. $p \in \mu^{-1}(\beta)$. Viceversa, if $p \in \mu^{-1}(\beta)$, then $||\mu(p)||^2$ is the maximum value of $f := ||\mu||^2$ and therefore $\hat{\beta}_p = 0$, where $\hat{\beta}$ denotes the Killing field on M induced by the element $\beta \in \mathfrak{k}$; moreover $\mu_\beta(p) = ||\beta||^2$ and therefore $p \in Z_\beta$. This implies that $\mu^{-1}(\beta)$ is a complex submanifold and that $C_\beta = K \cdot \mu^{-1}(\beta)$.

If S_β denotes the Morse stratum of C_β, we claim that $S_\beta = C_\beta$. Indeed, if $\gamma_t(q)$ denotes the flow of $-grad(f)$ through a point q belonging to the stratum S_β, then $\gamma_t(q)$ has a limit point in the critical subset C_β; since $f(\gamma_t(q))$ is non-increasing for $t \ge 0$ and $f(C_\beta)$ is the maximum value of f, we see that $f(\gamma_t(q)) = ||\beta||^2$ for all $t \ge 0$, that is $S_\beta \subseteq C_\beta$ and therefore $S_\beta = C_\beta$.

This implies that $C_\beta = S_\beta$ is a smooth complex submanifold of M and for every $y \in \mu^{-1}(\beta)$, we have

$$T_y S_\beta = T_y(K \cdot y) + T_y(\mu^{-1}(\beta)).$$

Now, if $v \in T_y(\mu^{-1}(\beta))$, then $v = Jw$ for some $w \in T_y(\mu^{-1}(\beta))$ and for every $X \in \mathfrak{k}$ we have

$$0 = \langle d\mu_y(w), X \rangle = \omega_y(w, \hat{X}_y) = \omega_y(Jv, \hat{X}_y) = g_y(v, \hat{X}_y),$$

meaning that $T_y(\mu^{-1}(\beta))$ is g-orthogonal to $T_y(K \cdot y)$. Since both S_β and $\mu^{-1}(\beta)$ are complex, this implies that $K \cdot y$ is a complex orbit. $\qquad \square$

We now give the proof of Theorem 1

Proof of Theorem 1. Assume f to be constant, i.e. the manifold M is mapped, by μ, into a sphere. We fix a point $x_o \in M$; we also recall that $\mu(M) \cap \mathfrak{t}_+^*$ is convex [K2], hence the manifold is mapped to a single coadjoint orbit $\mathcal{O} = K/K_\beta$, where $\beta = \mu(x_o)$. We then have that all the points of M are critical for f and $M = K \cdot \mu^{-1}(\beta)$ by the arguments used in the proof of Proposition 2.

Note that each K-orbit intersects $\mu^{-1}(\beta)$ in a single point. Indeed, if there are two points x and $z = k \cdot x$ for $k \in K$ which lie in $\mu^{-1}(\beta)$, then, by the K-equivariance of μ, we have $k \in K_\beta$ which is equal to K_x, since, for Proposition 2, the $K \cdot x$ orbit is complex; hence $k \cdot x = x$.

We can also prove that, for each $x \in M$, the tangent space $T_x K \cdot x$ is orthogonal to $T_x \mu^{-1}(\mu(x))$; indeed if $X \in T_x K \cdot x$ and $Y \in T_x \mu^{-1}(\mu(x))$, then, using the fundamental property of the moment map μ, we argue that $0 = \omega(Y, X) = g(Y, JX)$, where $JX \in T_x K \cdot x$ since the orbit is complex.

From this it follows that the map

$$\varphi : K/K_\beta \times \mu^{-1}(\beta) \to M, \qquad \varphi(gK_\beta, x) = g \cdot x,$$

where we identify K/K_β with the orbit $K \cdot x_o$, is a well defined K-equivariant diffeomorphism.

We also observe that $\mu^{-1}(\mu(x))$ is connected for all $x \in M$; moreover all the K-orbits are principal since their stabilizers are all equal to K_β, hence we have that K_x acts trivially on $T_x \mu^{-1}(\mu(x)) = (T_x K \cdot x)^\perp$ for all $x \in M$.

We now denote by $\mathcal{F}$ the foliation given by the K-orbits and by $\mathcal{F}^\perp$ the orthogonal foliation, so that $\mathcal{F}_y^\perp = T_y(\mu^{-1}(\mu(y)))$. We now claim that both $\mathcal{F}$ and $\mathcal{F}^\perp$ are totally geodesic; this will then imply that they are both parallel with respect to the Levi Civita connection and our result will follow.

We first observe that $\mathcal{F}^\perp$ is totally geodesic. Indeed, at each point $y \in M$, the stabilizer K_y is the centralizer of some torus in K and therefore its isotropy representation on the tangent space $T_y(K \cdot y)$ has no fixed vector; on the other hand, y is a principal point, so that the isotropy representation of K_y leaves $\mathcal{F}_y^\perp$ pointwisely fixed. This shows that $\mu^{-1}(\mu(y))$ is a connected component of the fixed point set of K_y and therefore it is totally geodesic.

We now claim that $\mathcal{F}$ is totally geodesic. We fix again a point $y \in M$ and a normal vector $\xi \in T_y(K \cdot y)^\perp$; the shape operator A_ξ of the orbit $K \cdot y$ relative to the normal vector ξ is a self-adjoint operator on $T_y(K \cdot y)$, which is K_y-invariant, since K_y leaves ξ fixed. We now decompose the Lie algebra $\mathfrak{k}$ of K as $\mathfrak{k} = \mathfrak{k}_\beta \oplus \mathfrak{m}$, where $\mathfrak{k}_\beta$ is the Lie algebra of K_β and $\mathfrak{m}$ is an $\mathrm{Ad}(K_\beta)$-invariant subspace which can be identified with the tangent space $T_y(K \cdot y)$. We also split $\mathfrak{m} = \bigoplus_{i=1}^l \mathfrak{m}_i$ as a sum of $\mathrm{Ad}(K_\beta)$-irreducible submodules; it is known that the summands $\mathfrak{m}_i$ are non-trivial and mutually inequivalent as $\mathrm{Ad}(K_\beta)$-modules (see e.g. [S]). This means that A_ξ preserves each $\mathfrak{m}_i$ and therefore, by Schur's Lemma, its restriction on each $\mathfrak{m}_i$ is a multiple of the identity. The complex structure J, viewed as a $\mathrm{Ad}(K_\beta)$-invariant

operator on $\mathfrak{m}$ also preserves each submodule and therefore it commutes with A_ξ. On the other hand, it is known (see e.g. [KN], p. 175) that the shape operator of a complex submanifold in a Kähler manifold anti-commutes with the complex structure; from this we conclude that $A_\xi = 0$ for all normal vectors ξ. This means that $\mathcal{F}$ is totally geodesic.

It is now easily seen that two orthogonal, integrable and totally geodesic foliations are parallel and this concludes the proof. $\qquad\square$

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